Section 3.10

Related Rates

by Joseph Phillip Brennan
Jila Niknejad

Related Rates Problems

In general we have been studying problems where one quantity is changing.

In a **related rates** problem, two or more related quantities are changing (often as functions of time).

- area and circumference of a circle
- radius, surface area, and volume of a sphere
- height of a kite, length of its cord, and angle of elevation
- ...

If two quantities are related, then so are their rates of change.

Example 1

A cube has side length x, and all its sides are growing at the same rate. How fast are the volume V and the surface area A changing?

$$V = x^3 \qquad A = 6x^2 \qquad \frac{dV}{dt} = 3x^2\frac{dx}{dt} \qquad \frac{dA}{dt} = 12x\frac{dx}{dt}$$

The rates of change of volume and surface area depend not only on the rate at which the side length changes, but also on the current side length.

$$\frac{\frac{dV}{dt}}{\frac{dA}{dt}} = \frac{3x^2\frac{dx}{dt}}{12x\frac{dx}{dt}} = \frac{x}{4}$$

Therefore, we can answer questions like this:

"If the volume is increasing at 4 cm^3/sec, how fast is the surface area increasing when the cube is 200 cm on each side?"

Related Rates Problems: The General Method

Solving Related Rates Problems

(1) If applicable, draw one or more figures representing the situation found in the problem.

(2) Identify the quantities in the problem. Clearly identify which are constants and which are variables.

(3) Determine which rates of change are known and which rates need to be calculated.

(4) Find an equation which **relates** the quantities whose rates you know to quantities whose rates you need to calculate.

- Often, this equation is geometric.

(5) Differentiate the equations implicitly and then substitute known quantities. Solve explicitly for the rates that need to be calculated.

Example 2: Oil is leaking from a hole in the ocean floor, forming a circular oil slick whose area is increasing at a constant rate of $5\,\mathrm{km}^2/\mathrm{day}$. How fast is the radius of the oil slick increasing when the area is $100\,\mathrm{km}^2$?

- We are interested in area A and radius r. Both are variable.
- We know $A' = \dfrac{dA}{dt} = 5\,\mathrm{km}^2/\mathrm{day}$. We want to find $r' = \dfrac{dr}{dt}$.
- The equation $A = \pi r^2$ relates the two quantities of interest.
- Differentiate with respect to time: $A' = 2\pi r\, r'$ or $\boxed{r' = \dfrac{A'}{2\pi r}.}$
- When $A = 100$ we have $r = \sqrt{A/\pi} = 10/\sqrt{\pi}$, so

$$r' = \frac{1}{2\pi}\frac{\sqrt{\pi}}{10}\cdot 5 = \frac{1}{4\sqrt{\pi}}\ \mathrm{km/day}.$$

- Note: The bigger A (thus r) gets, the <u>smaller</u> r' will be.

Example 2: The height of a cylinder is increasing at 6 meters per second and the radius is decreasing at 3 meters per second. How fast is the volume of the cylinder changing when the cylinder is 5 meters high and has a radius of 6 meters?

Solution:

- We are interested in three variables: volume V, radius R, height H.
- We know $\dfrac{dH}{dt} = 6\,\text{m/s}$ and $\dfrac{dR}{dt} = -3\,\text{m/s}$. We are searching for $\dfrac{dV}{dt}$.
- The volume equation $V = \pi R^2 H$ relates the quantities.
- Differentiate implicitly with respect to t:

$$\frac{dV}{dt} \;=\; 2\pi RH\frac{dR}{dt} + \pi R^2\frac{dH}{dt} \;=\; \boxed{36\pi\,\text{m}^3/\text{s}} \;\approx\; 113.10\,\text{m}^3/\text{s}$$

Example 3: Air is pumped into a spherical balloon at a constant rate of 3 cubic inches per second. When the radius of the balloon is 5 inches, how fast is the surface area expanding?

Solution:

- Variables of interest: radius R, volume V, and surface area S.
- We know $\dfrac{dV}{dt} = 3\,\text{in}^3/\text{s}$ but do not know $\dfrac{dR}{dt}$. We want to find $\dfrac{dS}{dt}$.
- The equations $V = \frac{4}{3}\pi R^3$ and $S = 4\pi R^2$ relate the quantities.
- Differentiate volume with respect to time t and solve for dR/dt:

$$\frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt} \qquad \Rightarrow \qquad \frac{dR}{dt} = \frac{3}{4\pi 5^2}\,\text{in/s}$$

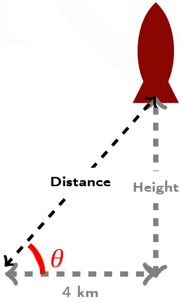
- Differentiate surface area with respect to time and plug in dR/dt:

$$\frac{dS}{dt} = 8\pi R \frac{dR}{dt} = \frac{6}{5}\,\text{in}^2/\text{s}$$

Example 4: You are filming a rocket launch in Florida and are standing 4 kilometers away from the launch pad.

(a) A few seconds after the launch, your camera is tilted at an angle of 30° above the horizon. If the angle is increasing at 9° per minute, how high is the rocket and how fast is it rising?

(b) When the rocket is 10 km high and rising at 8 km/s, how fast is the camera angle increasing?



Quantities of interest:

Height of rocket	H (variable)
Angle of camera	θ (variable)
Distance camera to rocket	D (variable)
Distance camera to launch pad	4 km (constant)

Example 4(a): Your camera is currently tilted at an angle of 30° above the horizon. If the angle is increasing at 9° per minute, how high is the rocket and how fast is it rising?

Solution: We know θ and $d\theta/dt$ and want to find H and dH/dt.

$$H = 4\tan(\theta) \qquad \frac{dH}{dt} = 4\sec^2(\theta)\frac{d\theta}{dt}$$

Currently $\theta = 30° = \pi/6$ and $d\theta/dt = 9° = \pi/20$. Plug in:

$$H = 4\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{4}{\sqrt{3}} \approx 2.309\,\text{km}}$$

$$\frac{dH}{dt} = 4\left(\frac{2}{\sqrt{3}}\right)^2 \frac{\pi}{20} = \boxed{\frac{4\pi}{15} \approx 0.838\,\text{km/min}}$$

Example 4(b): When the rocket is 10 km high and rising at 8 km/s, how fast is the camera angle increasing?

Solution: Now we know H and dH/dt and want to find $d\theta/dt$.

$$H = 4\tan(\theta) \qquad \frac{dH}{dt} = 4\sec^2(\theta)\frac{d\theta}{dt} \qquad \frac{d\theta}{dt} = \frac{\cos^2(\theta)}{4}\frac{dH}{dt}$$

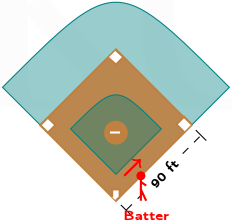
First we need to calculate $\cos(\theta)$. Use the Pythagorean Theorem:

$$\cos(\theta) = \frac{4}{D} = \frac{4}{\sqrt{4^2+10^2}} = \frac{2}{\sqrt{29}}$$

Now plug $\cos(\theta)$ and dH/dt into the formula for $d\theta/dt$:

$$\frac{d\theta}{dt} = \frac{4/29}{4}\cdot 8 = \boxed{\mathbf{\frac{8}{29}} \approx 0.28\,\text{radians/sec} \approx 15.8^\circ/\text{sec}}$$

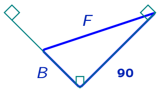
Example 5(a): A baseball diamond measures 90 feet on each side. A batter hits a ball along the third base line and runs towards first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling with horizontal speed 100 ft/s?



- Variables of interest: $B =$ distance the ball travels, $F =$ distance from ball to first base
- We know $B = 45\,\text{ft}$ and $\dfrac{dB}{dt} = 100\,\text{ft/s}$.
- We are searching for $\dfrac{dF}{dt}$.

The quantities B and F can be related using the Pythagorean Theorem.

Right now, $B = 45\,\text{ft}$ and $dB/dt = 100\,\text{ft/s}$. We are searching for dF/dt.



$$F^2 = B^2 + 90^2 \qquad (*)$$

Step 1: Use $(*)$ to find $F = \sqrt{45^2 + 90^2} = 45\sqrt{5}\,\text{ft}$

Step 2: Differentiate both sides of $(*)$ and solve for dF/dt.
Warning: Do not plug in the value of F before differentiating!

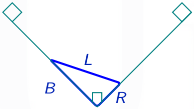
$$2F\frac{dF}{dt} = 2B\frac{dB}{dt} \qquad\qquad \frac{dF}{dt} = \frac{B}{F}\frac{dB}{dt}$$

Step 3: Plug in known values to find dF/dt.

$$\frac{dF}{dt} = \frac{B}{F}\frac{dB}{dt} = \frac{45\,\text{ft}}{45\sqrt{5}\,\text{ft}}(100\,\text{ft/s}) = 20\sqrt{5} \approx 44.7\,\text{ft/s}$$

Example 5(b): At the instant the batter has run $\frac{1}{8}$ of the way to first base, his instantaneous speed is 30 ft/s. At what rate is the distance between the ball and the batter changing?

- Variables of interest: B = distance ball has traveled; R = distance batter has run; L = distance from ball to runner.



Pythagorean Theorem:

$$L^2 = B^2 + R^2 \qquad (**)$$

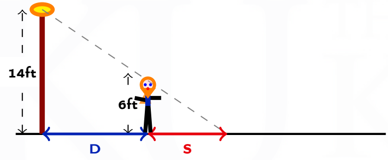
- $B = 45$ ft and $R = 90/8$ ft $\quad \therefore \quad L = \sqrt{B^2 + R^2} = \dfrac{45\sqrt{17}}{4}$ ft.
- Also, $dB/dt = 100$ ft/s and $dR/dt = 30$ ft/s.
- To find dL/dt, differentiate equation $(**)$:

$$2L\frac{dL}{dt} = 2B\frac{dB}{dt} + 2R\frac{dR}{dt} \qquad\qquad \frac{dL}{dt} = \frac{B\frac{dB}{dt} + R\frac{dR}{dt}}{L} \approx 104.3\,\text{ft/s}$$

Example 6: Joe, who is six feet tall, is walking away from a 14-foot tall lamppost at a constant rate of 3 feet per second. When Joe is 10 feet from the lamppost:

(a) How fast is the length of Joe's shadow changing?

(b) How fast is the tip of the shadow moving away from the lamp post?



- We are interested in two variables:
  D = Joe's distance from the lamppost
  S = length of Joe's shadow

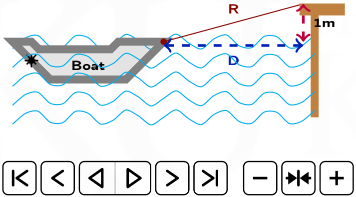
The quantities D and S are related via similar triangles:

$$\frac{D+S}{14} = \frac{S}{6} \qquad S = \frac{3}{4}D \qquad \frac{dS}{dt} = \frac{3}{4}\frac{dD}{dt}$$

Note that these rates of change do not depend on D!

$$(a)\ \frac{dS}{dt} = 2.25\,\mathrm{ft/s} \qquad (b)\ \frac{d}{dt}(D+S) = \frac{dD}{dt} + \frac{dS}{dt} = 5.25\,\mathrm{ft/s}$$

Example 7: A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 meter higher than the bow. If the rope is pulled at a rate of 1 meter per second, how fast is the boat approaching the dock when it is 8 meters from the dock?



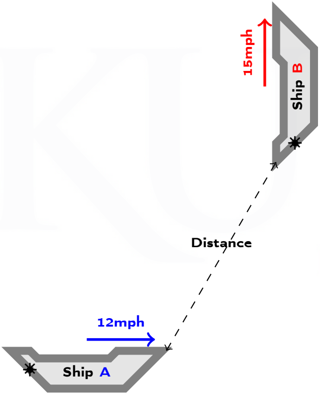
- The boat is constantly 1 meter below the dock. We are interested in two variables: the length of the rope between the bow and the pulley R, and the horizontal distance between the boat and the dock D.

- We know $\frac{dR}{dt} = -1\,\frac{m}{s}$ and we are searching for $\frac{dD}{dt}$.

- The quantities are related by the Pythagorean Theorem: $R^2 = 1^2 + D^2$.

$$\frac{dD}{dt} = \frac{R}{D}\frac{dR}{dt} = -\frac{\sqrt{65}}{8} \approx -1.008\,\frac{m}{s}$$

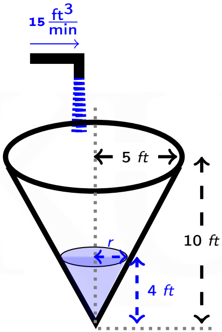
Example 8: At a certain moment, ship A is 6 miles south and 8 miles west of ship B. Ship A at that moment is steaming east at 12 mph, while ship B is steaming north at 15 mph. Are the ships approaching each other or separating from each other? At what rate?



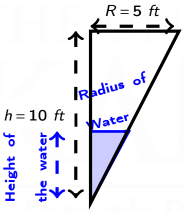
- Think of ship A as moving along the x-axis and ship B along the y-axis.
- Variables:
  X = difference in x-coordinates
  Y = difference in y-coordinates
  D = distance between ships
- $\dfrac{dY}{dt} = 15$ mph and $\dfrac{dX}{dt} = -12$ mph
- $D^2 = Y^2 + X^2$

$$\frac{dD}{dt} = \frac{Y}{D}\frac{dY}{dt} + \frac{X}{D}\frac{dX}{dt} = \frac{6}{10}(15) + \frac{8}{10}(-12) = \frac{-3}{5} \text{ mph}$$

Example 9(a): Water is being pumped at a rate of 15 cubic feet per minute into a conical reservoir 10 feet deep with a top radius of 5 feet. *How fast is the water level rising when the water is 4 feet deep?*



The cross section



$$\frac{\textbf{Radius of Water}}{\textbf{Height of Water}} = \frac{5\,ft}{10\,ft}$$

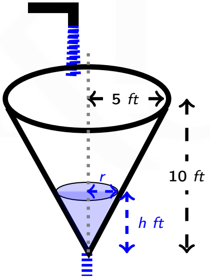
$$R = \frac{1}{2}H \qquad \frac{dR}{dt} = \frac{1}{2}\frac{dH}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{4}H^2\frac{dH}{dt}$$

- Three variables: volume of water V, height of water H, and radius R.
- We have $V'(t) = 15\,\text{ft}^3/\text{min}$ and are searching for $H'(t)$.
- By similar triangles, $R = H/2$ always. So $V = \frac{\pi}{3}R^2H$ becomes

$$V = \frac{\pi H^3}{12} \qquad V' = \frac{\pi H^2 H'}{4} \qquad H' = \frac{4V'}{\pi H^2} = \frac{15}{4\pi} \approx 1.194\,\text{ft/min}$$

Example 9(b): Water is being pumped at a rate of 15 cubic feet per minute into a leaky conical reservoir 10 feet deep with a top radius of 5 feet. Suppose that the water level is only rising at 1 foot/min when the water is 4 feet deep. *How fast is the water leaking out of the reservoir?*



- As before, $\dfrac{dV}{dt} = \dfrac{\pi H^2}{4}\dfrac{dH}{dt}$.
- Also, $\dfrac{dV}{dt} = \text{pumping rate} - \text{leakage rate}$.

$$\text{leakage rate} = \text{pumping rate} - \frac{\pi H^2}{4}\frac{dH}{dt}$$

$$= \boxed{15 - \frac{16\pi}{4} \approx 2.44\ \text{ft}^3/\text{min}}$$

Example 9(c): A conical pile of gravel has volume $2000\,\mathrm{m}^3$. The pile is gradually collapsing, becoming shorter and wider; its height is decreasing at a constant rate of 4 m/hr.

If the pile is currently 20 m high, how fast is the radius increasing?

Solution #1: We know that

$$V = 2000 = \frac{1}{3}\pi r^2 h. \qquad (*)$$

Differentiate both sides with respect to time:

$$0 = \frac{\pi}{3}(2rr'h + r^2h').$$

Plug in $h' = -4\,\mathrm{m/sec}$ and solve for r':

$$r' = \frac{2r}{h}\ \mathrm{m/sec} \qquad (**)$$

Plug $h = 20$ into $(*)$ to solve for r, then use $(**)$:

$$r = \sqrt{\frac{300}{\pi}} \qquad\qquad r' = \frac{2\sqrt{300/\pi}}{20} = \frac{\sqrt{3}}{\sqrt{\pi}} \approx 0.977\,\mathrm{m/hr}$$

Solution #2: First solve $(*)$ for r to get

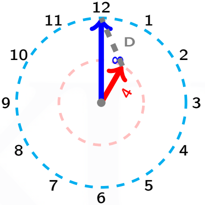
$$r = \sqrt{\frac{6000}{\pi h}} = \frac{20\sqrt{15}}{\sqrt{\pi}} h^{-1/2}$$

then differentiate both sides with respect to time to get

$$\frac{dr}{dt} = \frac{-10\sqrt{15}}{\sqrt{\pi}} h^{-3/2} \frac{dh}{dt}$$

and now plug in $h = 20$, $h' = -4$ to again get $r' = \dfrac{\sqrt{3}}{\sqrt{\pi}} \approx 0.977\,\text{m/hr}$.

Example 10: The minute hand on a watch is 8 millimeters long and the hour hand is 4 millimeters long. How fast is the distance between the tips of the hands changing at one o'clock?



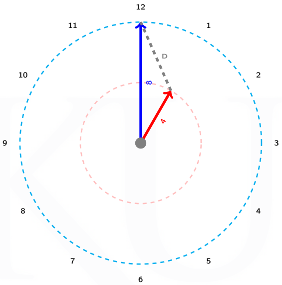
- Two constants are the lengths of the hands $M = 8\,\text{mm}$ and $H = 4\,\text{mm}$. Two variables are the distance between the hands D and the angle θ.

- We are searching for $\frac{dD}{dt}$. We can calculate $\frac{d\theta}{dt}$ as the difference in turning speed of the hour and minute hand (respectively as both hands turn clockwise).

$$\frac{d\theta_H}{dt} - \frac{d\theta_M}{dt} = \frac{2\pi}{(12)(60)} - \frac{2\pi}{60} = \frac{-11\pi}{360}\,\frac{\text{radians}}{\text{minute}}$$

- The quantities are related by the Law of Cosines: $D^2 = M^2 + H^2 - 2MH\cos(\theta)$

The minute hand on a watch is 8 millimeters long and the hour hand is 4 millimeters long. How fast is the distance between the tips of the hands changing at one o'clock?



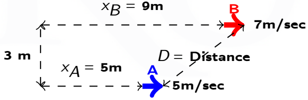
- $M = 8\,\text{mm},\; H = 4\,\text{mm},$
  $\dfrac{d\theta}{dt} = \dfrac{-11\pi}{360}\,\dfrac{\text{radians}}{\text{minute}}$

$$D^2 = M^2 + H^2 - 2MH\cos(\theta)$$

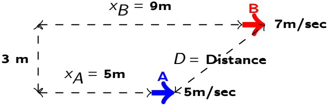
$$\theta = \frac{\pi}{6} \qquad D = \sqrt{8^2 + 4^2 - 2(8)(4)\cos\left(\frac{\pi}{6}\right)} = 4\sqrt{5 - 2\sqrt{3}}$$

$$\frac{dD}{dt} = \frac{MH}{D}\sin(\theta)\frac{d\theta}{dt} = \frac{(8)(4)}{4\sqrt{5-2\sqrt{3}}}\sin\left(\frac{\pi}{6}\right)\frac{-11\pi}{360} \approx -0.310\text{mm/min}$$

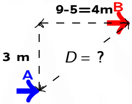
Example 11: Two arrows, arrow A and arrow B, have been shot by two people standing 3 meters apart by a wall in the perpendicular direction to the wall at the same time. At the instant that arrow A has traveled 5 meters, arrow B has traveled 9 meters. At that instant, arrow A is traveling at 5 meters per second and arrow B is traveling at 7 meters per second. How fast is their distance changing at that instant?

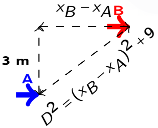


- The distance traveled for each arrow, x_A, x_B, and the distance between them, D, is changing.
- The distance between the paths that they are traveling on is constant, 3 m.



- To find the distance at that instance, form a right triangle as shown in the picture. Then use Pythagorean Theorem: $D^2 = 3^2 + 4^2 \implies D = 5$





- Pythagorean theorem for variables: $D^2 = (x_B - x_A)^2 + 9$.
- Differentiate: $2D\frac{dD}{dt} = 2(x_B - x_A)\left(\frac{dx_B}{dt} - \frac{dx_A}{dt}\right)$.
- Replace the values and solve: $2(5)\frac{dD}{dt} = 2(4)(7-5)$.

$$\implies \boxed{\frac{dD}{dt} = \frac{12}{5}}$$